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Existence of sign-changing solution with least energy for a class of Kirchhoff-type equations in $\mathbb{R}^N$

Xianzhong Yao ✉ [1, 2] and **Chunlai Mu** [2]

[1] Faculty of Applied Mathematics, Shanxi University of Finance and Economics
Taiyuan 030006, P.R. China
[2] College of Mathematics and Statistics, Chongqing University, Chongqing 401331, P.R. China

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Abstract. We consider the existence of least energy sign-changing (nodal) solution of Kirchhoff-type elliptic problems with general nonlinearity. Using a truncated technique and constrained minimization on the nodal Nehari manifold, we obtain that the Kirchhoff-type elliptic problem possesses one least energy sign-changing solution by applying a Pohožaev type identity. Moreover, the energy of the sign-changing solution is strictly more than the ground state energy.

Keywords: Kirchhoff-type, ground state solution, sign-changing solution, Pohožaev type identity.

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1 Introduction

In this paper, we are concerned with the following Kirchhoff-type elliptic problem with general nonlinearity:

$$\left(a + \lambda \int_{\mathbb{R}^N} |\nabla u|^2 dx + \lambda b \int_{\mathbb{R}^N} u^2 dx\right) [-\Delta u + bu] = f(u), \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $a, b > 0$ are constants, $\lambda > 0$ is a parameter and $N \geq 3$. Moreover, $f \in C^1(\mathbb{R}, \mathbb{R}^+)$ satisfies the following hypotheses:

(f_1) $|f(t)| \leq C(|t| + |t|^{q-1})$ for $q \in (2, 2^*)$, $2^* = \frac{2N}{N-2}$;

(f_2) $f(t) = o(|t|)$ as $t \to 0$;

(f_3) $\lim\limits_{t\to\infty} \frac{f(t)}{|t|} = +\infty$;

(f_4) $\frac{f(t)}{|t|}$ is strictly increasing in $\mathbb{R}\backslash\{0\}$.

✉ Corresponding author. Email: yaoxz@sxufe.edu.cn

Kirchhoff-type problems are often referred to as being nonlocal because of the presence of the integral terms. It is related to the stationary analogue of the equation that arise in the study of string or membrane vibrations, namely

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L |\frac{\partial u}{\partial x}|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0, \tag{1.2}$$

which was presented by Kirchhoff [10] in 1883. This model is an extension of the classical d'Alembert wave equation by considering the effects of the changes on the length of the elastic string during the free vibrations. The parameters in the Kirchhoff's model have the following meanings: L is the length of the string, h is the area of cross-section, E is the Young modulus of the material, ρ is the mass density and P_0 is the initial tension. Some early classical studies of Kirchhoff-type equations were those of Pohožaev [22] and Bernstein [3]. However, Kirchhoff's model received great attention only after Lions [13] proposed following abstract framework for the model (1.2),

$$\begin{cases} u_{tt} - (a + b \int_\Omega |\nabla u|^2 dx)\Delta u = f(x,u), & x \in \Omega, \\ u = 0 & x \in \partial\Omega. \end{cases} \tag{1.3}$$

The existence and concentration behavior of solutions to Kirchhoff-type elliptic problem have been extensively studied in the past decade. Most researchers paid their attention to focus on existence of positive solutions, ground state, radial and nonradial solutions and semiclassical state under some different assumptions, see for example [1,4,6,7,11,12,17,19–21,24,26] and references therein. While existence of sign-changing solutions has been received few attention, and there are very few results on existence of sign-changing solutions to Kirchhoff-type problem. Only Zhang et al. [18,28] investigated the existence of sign-changing solution of the Kirchhoff-type problem (1.4),

$$\begin{cases} -(a + b \int_\Omega |\nabla u|^2 dx)\Delta u = f(u), & x \in \Omega, \\ u = 0 & x \in \partial\Omega, \end{cases} \tag{1.4}$$

where $a > 0$, $b \geq 0$ and $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a bounded domain with smooth boundary. By using variational methods and invariant sets of descent flow, they demonstrated that equations (1.4) possesses a sign-changing solution with nonlinearity f satisfying some suitable conditions.

In recent years, there has been increasing attention to the existence of sign-changing (nodal) solutions to Kirchhoff-type problem. In [23], Shuai considered equations (1.4) in $N = 1,2,3$ with $f \in C^1(\mathbb{R},\mathbb{R})$ satisfying following conditions:

(H_1) $f(t) = o(|t|)$ as $t \to 0$;

(H_2) for some constant $p \in (4, 2^*)$, $\lim\limits_{t\to\infty} \frac{f(t)}{t^{p-1}} = 0$, where $2^* = +\infty$ for $N = 1,2$, if $N = 3$, $2^* = 6$;

(H_3) $\lim\limits_{t\to\infty} \frac{F(t)}{t^4} = +\infty$, where $F(t) = \int_0^t f(s)ds$;

(H_4) $\frac{f(t)}{|t|^3}$ is an increasing function in $\mathbb{R}\backslash\{0\}$.

Employing constraint variational method and quantitative deformation lemma, author asserted that there is one least energy sign-changing solution (nodal solution), which has precisely two nodal domains. Moreover, the energy of sign-changing solution is strictly larger than the ground state energy. While Figueiredo and Nascimento in [5] discussed the following more general problem than (1.4), for $N = 3$,

$$\begin{cases} -M\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u = f(u), & x \in \Omega, \\ u = 0 & x \in \partial\Omega, \end{cases} \tag{1.5}$$

where $M, f \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ fulfill some assumptions:

(M_1) function M is increasing and $M(0) := m_0 > 0$;

(M_2) $\frac{M(t)}{t}$ is a decreasing function for $t > 0$;

$(\widetilde{H}_3)$ there is $\theta \in (4, 6)$ such that $0 < \theta F(t) \leq f(t)t$, for $t \neq 0$.

Under the conditions (M_1), (M_2) and (H_1), (H_2), $(\widetilde{H}_3)$, (H_4), they explored that there exists one least energy nodal solution to the problem (1.5). For more results, we refer to [2, 16, 27] for some variant version of Kirchhoff-type problem.

From the discussion above, we discover that researchers usually need suppose that f satisfies (H_4) and (H_3) or $(\widetilde{H}_3)$, which ensure the boundedness of a minimum sequence for the corresponding functional of the Kirchhoff-type problem. As well it also guarantees that the nodal Nehari manifold of corresponding functional of the Kirchhoff-type problem is not empty. Then their results can be derived by usual variational methods and quantitative deformation lemma. In this paper, we replace the conditions (H_4) and (H_3) or $(\widetilde{H}_3)$ by the hypotheses (f_4) and (f_3), which is weaker than the conditions in foregoing literatures. A typical case is that $f(u) = |u|^{p-1}u$ for $p \in (1, 5)$, however, the results in the references above is valid only for $p \in (3, 5)$. To the best authors' knowledge, there is no result on the existence of least energy sign-changing (nodal) solution to Kirchhoff-type problem with nonlinearity f satisfying the hypotheses (f_3) and (f_4).

To character our results, we need first to introduce the energy functional for corresponding Kirchhoff-type problem (1.1) and nodal Nehari manifold. Let $H^1(\mathbb{R}^N)$ be the usual Sobolev space equipped with the inner product and norm

$$(u, v) = \int_{\mathbb{R}^3} \nabla u \nabla v + buv dx, \quad \|u\| = (u, u)^{1/2},$$

and $L^p(\mathbb{R}^N)$ is the usual Lebesgue space endowed with the norm

$$|u|_p = \left(\int_{\mathbb{R}^N} |u|^p dx\right)^{1/p}, \quad \text{for } 1 \leq p < \infty, \quad |u|_\infty = \sup_{x \in \mathbb{R}^N} |u(x)|,$$

as well as

$$\mathcal{D}^{1,2}(\mathbb{R}^N) := \{u \in L^{2^*}(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N)\}$$

with norm $\|u\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)} = |\nabla u|_2$. It is well known that the embedding of $H^1(\mathbb{R}^N)$ into $L^p(\mathbb{R}^N)$ for $p \in [2, 2^*]$ is continuous but not compact. Denote the subspace $H^1_r(\mathbb{R}^N) := \{u \in H^1(\mathbb{R}^N) : u \text{ is radial symmetric function}\}$ and hereafter, for simplicity, $H := H^1_r(\mathbb{R}^N)$. Then $H \hookrightarrow L^p(\mathbb{R}^N)$ compactly for $p \in (2, 2^*)$, see [25, Corollary 1.26].

Define the energy functional associated with equation (1.1), $J_\lambda : H \to \mathbb{R}$ given by

$$J_\lambda(u) = \frac{a}{2}\|u\|^2 + \frac{\lambda}{4}\|u\|^4 - \int_{\mathbb{R}^N} F(u)dx.$$

Obviously, J_λ belong to $C^1(H, \mathbb{R})$. For any $u, v \in H$, there is

$$\langle J'_\lambda(u), v\rangle = a(u, v) + \lambda\|u\|^2(u, v) - \int_{\mathbb{R}^N} f(u)vdx.$$

It is well-known that each weak solution of equation (1.1) corresponds a critical point of J_λ. We define the Nehari manifold for the corresponding energy functional J_λ

$$\mathcal{N}_\lambda = \{u \in H\backslash\{0\} : \langle J'_\lambda(u), u\rangle = 0\},$$

and the nodal Nehari manifold

$$\mathcal{M}_\lambda = \{u \in H : u^\pm \neq 0, \langle J'_\lambda(u), u^\pm\rangle = 0\},$$

where

$$u^+(x) = \max\{u(x), 0\} \quad \text{and} \quad u^-(x) = \min\{u(x), 0\}.$$

Moreover, denote

$$\widetilde{c}_\lambda := \inf\{J_\lambda(u) : u \in \mathcal{N}_\lambda\} \quad \text{and} \quad c_\lambda := \inf\{J_\lambda(u) : u \in \mathcal{M}_\lambda\}.$$

When u is a nontrivial solution to equation (1.1) and $J_\lambda(u) \leq J_\lambda(v)$, where v is any solution of equation (1.1), then we say that $u \in H$ is a ground state (least energy) solution to equation (1.1) and u is one sign-changing (nodal) solution to equation (1.1) if $u^\pm \neq 0$. By Lemma 2.3 below, we have that $\mathcal{N}_\lambda$ and $\mathcal{M}_\lambda$ are not empty and $\mathcal{M}_\lambda \subset \mathcal{N}_\lambda$. From the definition of $\mathcal{N}_\lambda$ and $\mathcal{M}_\lambda$, we know that all nontrivial solutions and sign-changing solutions to equation (1.1) are included in $\mathcal{N}_\lambda$ and $\mathcal{M}_\lambda$, respectively.

Now, we give our main results as follows.

Theorem 1.1. *Assume the conditions (f_1)–(f_4) hold. Then there exists a positive Λ such that, for any $\lambda \in (0, \Lambda)$, the problem (1.1) have a ground state solution u_λ which is constant sign and a least energy sign-changing solution v_λ satisfying*

$$c_\lambda = J_\lambda(v_\lambda) > J_\lambda(u_\lambda) = \widetilde{c}_\lambda > 0.$$

The remainder of this paper is organized as follows. In Section 2, we present the abstract framework of the problem as well as some preliminary results. Theorem 1.1 will be proved in Section 3.

2 Preliminaries

In this section, we show examples how theorems, definitions, lists and formulae should be formatted.

In this section, we give some notations and lemmas. According to the foregoing discussion, we know that it is very difficult to obtain bounded minimum sequences for the associated functional J_λ. So we here use a truncated technique, following [8, 9, 11], to handle it. We introduce a cut-off function $\phi \in C^\infty(\mathbb{R}, \mathbb{R})$ satisfying

$$\begin{cases} \phi(t) = 1, & t \in [0,1], \\ 0 \leq \phi(t) \leq 1, & t \in (1,2), \\ \phi(t) = 0, & t \in [2,\infty), \\ |\phi'|_\infty \leq 2, \end{cases}$$

and then consider the following truncated functional $J_{\lambda,\kappa} : H \to \mathbb{R}$ defined by

$$J_{\lambda,\kappa}(u) = \frac{a}{2}\|u\|^2 + \frac{\lambda}{4}h_\kappa(u)\|u\|^4 - \int_{\mathbb{R}^N} F(u)dx,$$

where for every $\kappa > 0$,

$$h_\kappa(u) = \phi\left(\frac{\|u\|^2}{\kappa^2}\right).$$

It is easy to know that $J_{\lambda,\kappa}$ belong to $\mathcal{C}^1(H, \mathbb{R})$. For $\kappa > 0$ enough large, we can take advantage of $J_{\lambda,\kappa}$ to obtain a critical point w_λ of $J_{\lambda,\kappa}$, then, by the definition of ϕ and $J_{\lambda,\kappa}$, we know that w_λ is a critical point of J_λ if we show that $\|w_\lambda\| \leq \kappa$. We define the Nehari manifold of $J_{\lambda,\kappa}$ as follows

$$\mathcal{N}_{\lambda,\kappa} = \{u \in H\backslash\{0\} : \langle J'_{\lambda,\kappa}(u), u\rangle = 0\}$$

and the nodal Nehari manifold

$$\mathcal{M}_{\lambda,\kappa} = \{u \in H : u^\pm \neq 0, \langle J'_{\lambda,\kappa}(u), u^\pm\rangle = 0\}.$$

Moreover, denote

$$\widetilde{c}_{\lambda,\kappa} := \inf\{J_{\lambda,\kappa}(u) : u \in \mathcal{N}_{\lambda,\kappa}\}, \qquad c_{\lambda,\kappa} := \inf\{J_{\lambda,\kappa}(u) : u \in \mathcal{M}_{\lambda,\kappa}\}.$$

Notation 2.1. Throughout this paper, we denote by "$\to$" and "$\rightharpoonup$" the strong and weak convergence in the related function space, respectively. $B_r(x) := \{y \in \mathbb{R}^N : |x-y| < r\}$. We use $o(1)$ to denote any quantity which tends to zero as $n \to \infty$. We will use the symbol C and C_i for denoting positive constants unless otherwise stated explicitly and the value of C and C_i is allowed to change from line to line and also in the same formula.

Lemma 2.2. *For all $u \in \mathcal{N}_{\lambda,\kappa}$, the following results hold:*

(i) *for any $\lambda > 0$, There exists $r > 0$ such that $\|u\| \geq r$;*

(ii) *$J_{\lambda,\kappa}$ has a lower bound in $\mathcal{N}_{\lambda,\kappa}$.*

Proof. For any $u \in \mathcal{N}_{\lambda,\kappa}$, there is

$$a\|u\|^2 + \lambda h_\kappa(u)\|u\|^4 + \frac{\lambda}{2\kappa^2}\phi'\left(\frac{\|u\|^2}{\kappa^2}\right)\|u\|^6 = \int_{\mathbb{R}^N} f(u)udx, \tag{2.1}$$

By (f_1), (f_2) and Sobolev's inequality, it is easy to obtain the result (i) if $\|u\|^2 \geq 2\kappa^2$, otherwise, the following inequality holds

$$a\|u\|^2 + \lambda h_\kappa(u)\|u\|^4 + \frac{\lambda}{2\kappa^2}\phi'\left(\frac{\|u\|^2}{\kappa^2}\right)\|u\|^6 \geq a\|u\|^2 - \frac{\lambda}{\kappa^2}\|u\|^6,$$

owing to (f_1) and (f_2), we have, for small $\varepsilon > 0$,

$$\int_{\mathbb{R}^N} f(u)udx \le \varepsilon|u|_2^2 + C_\varepsilon|u|_q^q. \tag{2.2}$$

Combining the three formulas above and Sobolev inequality, we obtain that

$$a\|u\|^2 - \frac{\lambda}{\kappa^2}\|u\|^6 \le \int_{\mathbb{R}^N} f(u)udx \le \varepsilon|u|_2^2 + C_\varepsilon|u|_q^q \le \varepsilon C_1\|u\|^2 + C_2\|u\|^q.$$

It follows the assertion (i).

Next we show the item (ii). If $\|u\|^2 \ge 2\kappa^2$ for all $u \in \mathcal{N}$, by the definition of ϕ, we observe

$$J_{\lambda,\kappa}(u) = \frac{a}{2}\|u\|^2 - \int_{\mathbb{R}^N} F(u)dx,$$

and by (2.1), it holds

$$a\|u\|^2 = \int_{\mathbb{R}^N} f(u)udx.$$

Since (f_4) implies that $2F(t) \le f(t)t$ for $t \in \mathbb{R}$, we deduce that $J_{\lambda,\kappa}(u) > 0$ and the result is finished. Suppose, by contradiction, that there is $u \in \mathcal{N}$ such that $\|u\|^2 < 2\kappa^2$. In which case, the result is valid by $J_{\lambda,\kappa} \in \mathcal{C}^1(H, \mathbb{R})$. Thus the conclusion is established. □

Lemma 2.3. *For any $u \in H$ with $u^\pm \neq 0$, then there is a pair $(t_u, s_u) \in \mathbb{R}^+ \times \mathbb{R}^+$ such that $t_u u^+ + s_u u^- \in \mathcal{M}_{\lambda,\kappa}$ for λ small. In particular, $\mathcal{M}_{\lambda,\kappa} \neq \emptyset$ and for all $(t,s) \in \mathbb{R}^+ \times \mathbb{R}^+$, there is*

$$J_{\lambda,\kappa}(t_u u^+ + s_u u^-) \ge J_{\lambda,\kappa}(tu^+ + su^-).$$

Proof. For any $u \in H$ with $u^\pm \neq 0$, define function $g : [0,\infty) \times [0,\infty) \to \mathbb{R}$ given by

$$g(t,s) := J_{\lambda,\kappa}(tu^+ + su^-)$$

and its gradient $\Phi : [0,\infty) \times [0,\infty) \to \mathbb{R} \times \mathbb{R}$, denoted by

$$\begin{aligned}\Phi(t,s) &:= \big(\Phi_1(t,s), \Phi_2(t,s)\big) = \Big(\frac{\partial g}{\partial t}(t,s), \frac{\partial g}{\partial s}(t,s)\Big) \\ &= \Big(\langle J'_{\lambda,\kappa}(tu^+ + su^-), u^+\rangle, \langle J'_{\lambda,\kappa}(tu^+ + su^-), u^-\rangle\Big).\end{aligned}$$

We simply compute, by $(f_1)(f_2)$ and Sobolev inequality,

$$\begin{aligned}g(t,s) &\ge \frac{at^2}{2}\|u^+\|^2 - \varepsilon t^2|u^+|_2^2 - Ct^q|u^+|_q^q + \frac{as^2}{2}\|u^-\|^2 - \varepsilon s^2|u^-|_2^2 - Cs^q|u^-|_q^q \\ &\ge \frac{at^2}{2}\|u^+\|^2 - \varepsilon C_1 t^2\|u^+\|^2 - C_2 t^q\|u^+\|^q + \frac{as^2}{2}\|u^-\|^2 - \varepsilon C_3 s^2\|u^-\|^2 - C_4 s^q\|u^-\|^q,\end{aligned}$$

for small $\varepsilon > 0$ and some positive constants C_i $(i = 1,2,3,4)$. Therefore, $g(t,s)$ is positive for (t,s) small. Since (f_3), for t large enough, there exists a large $M > 0$ such that

$$f(t) \ge M|t|. \tag{2.3}$$

Thus, for (t,s) large enough, we compute

$$\begin{aligned}g(t,s) &= J_{\lambda,\kappa}(tu^+ + su^-) \\ &= \frac{a}{2}t^2\|u^+\|^2 + \frac{a}{2}s^2\|u^-\|^2 + \frac{\lambda}{4}h_\kappa(tu^+ + su^-)\|tu^+ + su^-\|^4 - \int_{\mathbb{R}^N} F(tu^+ + su^-)dx \\ &= \frac{a}{2}t^2\|u^+\|^2 + \frac{a}{2}s^2\|u^-\|^2 - \int_{\mathbb{R}^N} F(tu^+) + F(su^-)dx \\ &\le \frac{a}{2}t^2\|u^+\|^2 + \frac{a}{2}s^2\|u^-\|^2 - Mt^2\int_{\mathbb{R}^N} |u^+|^2dx - Ms^2\int_{\mathbb{R}^N} |u^-|^2dx,\end{aligned} \tag{2.4}$$

therefore, for (t,s) large enough, we have $g(t,s) \to -\infty$. So there is a pair of (t_u, s_u) such that

$$g(t_u, s_u) = \max_{t,s\geq 0} g(t,s).$$

We next claim that $t_u, s_u > 0$. Indeed, without loss of generality, assuming the pair of $(t_u, 0)$ is a maximum point of $g(t,s)$, we get that

$$\begin{aligned}
\frac{\partial}{\partial s} g(t_u, s) &= as\|u^-\|^2 + \frac{\lambda}{4} h_\kappa(t_u u^+ + su^-)(4t_u^2 s\|u^+\|\|u^-\|^2 + 4s^3\|u^-\|^2) \\
&\quad + \frac{\lambda s}{2\kappa^2} h'_\kappa(t_u u^+ + su^-)\|t_u u^+ + su^-\|^4\|u^-\|^2 - \int_{\mathbb{R}^N} f(su^-)u^- dx \qquad (2.5) \\
&\geq as\|u^-\|^2 - \int_{\mathbb{R}^N} f(su^-)u^- dx - \frac{\lambda s}{\kappa^2}\|t_u u^+ + su^-\|^4\|u^-\|^2,
\end{aligned}$$

since condition (f_2), for λ, s enough small, we see that $\frac{\partial}{\partial s} g(t_u, s) > 0$, which implies that $g(t_u, s)$ is increasing for s small. This contradicts that the pair of $(t_u, 0)$ is a maximum point of $g(t,s)$. Consequently, (t_u, s_u) is a positive maximum point of $g(t,s)$.

Finally, we prove that $t_u u^+ + s_u u^- \in \mathcal{M}_{\lambda,\kappa}$. According to the definition of Φ, we note that $t_u u^+ + s_u u^- \in \mathcal{M}_{\lambda,\kappa}$ is equivalent to $\Phi(t,s) = 0$ for any $t, s > 0$. Because the pair of (t_u, s_u) is a positive maximum point of $g(t,s)$, we observe that

$$\frac{\partial}{\partial t} g(t,s)|_{(t_u,s_u)} = \frac{\partial}{\partial s} g(t,s)|_{(t_u,s_u)} = 0,$$

is equal to

$$\langle J'_{\lambda,\kappa}(t_u u^+ + s_u u^-), u^+\rangle = \langle J'_{\lambda,\kappa}(t_u u^+ + s_u u^-), u^-\rangle = 0,$$

which is same as

$$\Phi(t_u, s_u) = 0.$$

Thence, by virtue of the definition of nodal Nehari manifolds, we show that $t_u u^+ + s_u u^- \in \mathcal{M}_{\lambda,\kappa}$, which finishes the proof. □

Corollary 2.4. *For any $u \in H\backslash\{0\}$, then there exists a $t_u \in \mathbb{R}^+$ such that $t_u u \in \mathcal{N}_{\lambda,\kappa}$ for λ small. In particular, $\mathcal{N}_{\lambda,\kappa} \neq \emptyset$ and for all $t \in \mathbb{R}^+$, there is*

$$J_{\lambda,\kappa}(t_u u) \geq J_{\lambda,\kappa}(tu).$$

Lemma 2.5 (see Lions [14,15])**.** *Let $r > 0$ and $p \in [2, 2^*)$. If $\{u_n\}$ is bounded in H and*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_r(y)} |u_n|^p dx = 0,$$

then we have $u_n \to 0$ in $L^q(\mathbb{R}^N)$ for $q \in (2, 2^*)$.

Lemma 2.6. *Let $\{u_n\} \subset \mathcal{N}_{\lambda,\kappa}$ be a minimum sequence of $J_{\lambda,\kappa}$ at level $\widetilde{c}_{\lambda,\kappa}$, then $\{u_n\}$ is bounded in H.*

Proof. Arguing by contradiction, suppose $\|u_n\| \to \infty$ as $n \to \infty$, and set $v_n := \frac{u_n}{\|u_n\|}$. Then there exists a $v \in H$ such that $v_n \rightharpoonup v$ in H, up to a subsequence. Moreover, for $p \in [2, 2^*)$, we have either $\{v_n\}$ is vanishing, i.e.,

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_r(y)} |v_n|^p dx = 0$$

or non-vanishing, i.e., there exist $r, \delta > 0$ and a sequence $\{y_n\} \subset \mathbb{R}^N$such that

$$\lim_{n\to\infty} \int_{B_r(y_n)} |v_n|^p dx \geq \delta > 0.$$

We next shall prove neither vanishing nor non-vanishing occurs and this will provide the desired contradiction. If $\{v_n\}$ is vanishing, by Lemma 2.5, this implies $v_n \to 0$ in $L^q(\mathbb{R}^N)$ for $q \in (2, 2^*)$. Then, for every $t > 0$, we have, in view of (f_1), (f_2) and Sobolev's inequality,

$$\begin{aligned}
\widetilde{c}_{\lambda,\kappa} + o(1) &= J_{\lambda,\kappa}(u_n) \geq J_{\lambda,\kappa}(tv_n) \\
&= \frac{at^2}{2}\|v_n\|^2 + \frac{\lambda}{4}h(v_n)\|v_n\|^4 - \int_{\mathbb{R}^N} F(tv_n)dx \\
&\geq \frac{at^2}{2} - \varepsilon t^2 \int_{\mathbb{R}^N} v_n^2 dx - C_\varepsilon t^q \int_{\mathbb{R}^N} |v_n|^q dx \\
&\geq \frac{at^2}{2} - \varepsilon C_1 t^2 - C_\varepsilon t^q \int_{\mathbb{R}^N} |v_n|^q dx \\
&\to \frac{at^2}{2} - \varepsilon C_1 t^2,
\end{aligned}$$

as $n \to \infty$. This yields a contradiction for enough large t .

Should non-vanishing occur, we then check that for enough large n, by (f_3)

$$\begin{aligned}
0 \leq \frac{J_{\lambda,\kappa}(u_n)}{\|u_n\|^2} &= \frac{a}{2} - \int_{\mathbb{R}^N} \frac{F(u_n)}{u_n^2}|v_n|^2 dx \\
&\leq \frac{a}{2} - \int_{|u_n|>M} \frac{F(u_n)}{u_n^2}|v_n|^2 dx - \int_{|u_n|\leq M} \frac{F(u_n)}{u_n^2}|v_n|^2 dx \\
&\leq \frac{a}{2} - M\int_{|u_n|>M} |v_n|^2 dx \\
&\leq \frac{a}{2} - M\int_{[|u_n|>M]\cap B_r(y_n)} |v_n|^2 dx \\
&\leq \frac{a}{2} - M\int_{B_r(y_n)} |v_n|^2 dx \\
&\leq \frac{a}{2} - M\delta \\
&< 0,
\end{aligned}$$

where M is enough large. This is a contradiction and completes the proof. □

Lemma 2.7. *Let $\{u_n\} \subset \mathcal{M}_{\lambda,\kappa}$ be a minimum sequence for $J_{\lambda,\kappa}$ at level $c_{\lambda,\kappa}$, then $\{u_n\}$ has a convergent subsequence in H.*

Proof. Let $\{u_n\} \subset \mathcal{M}_{\lambda,\kappa}$ be such that

$$J_{\lambda,\kappa}(u_n) \to c_{\lambda,\kappa}, \quad \text{as } n \to \infty.$$

Then, by Lemma 2.6, we know that u_n is bounded in H and there exists a $u \in H$, up to a subsequence, such that

$$\begin{aligned}
u_n &\rightharpoonup u \quad \text{in } H, \\
u_n &\to u \quad \text{in } L^p(\mathbb{R}^N) \text{ for } p \in (2, 2^*), \\
u_n &\to u \quad \text{a.e. in } \mathbb{R}^N.
\end{aligned} \tag{2.6}$$

From (f_1) and (f_2), we have, for ε small,

$$|f(t)| \le \varepsilon|t| + C_\varepsilon |t|^{q-1}, \quad \text{for any } t \in \mathbb{R}, \tag{2.7}$$

thus by Hölder's inequality and Sobolev's inequality, we get

$$\begin{aligned} \left| \int_{\mathbb{R}^N} f(u_n)(u_n - u)dx \right| &\le \varepsilon |u_n|_2 |u_n - u|_2 + C_\varepsilon \int_{\mathbb{R}^N} |u_n|^{q-1}|u_n - u|dx \\ &\le \varepsilon |u_n|_2 |u_n - u|_2 + C_\varepsilon |u_n|_q^{q-1} |u_n - u|_q. \end{aligned} \tag{2.8}$$

Thus thanks to boundedness of $\{u_n\}$ in H and (2.6), we obtain that

$$\int_{\mathbb{R}^N} f(u_n)(u_n - u)dx \to 0 \quad \text{as } n \to \infty.$$

Then note that for n enough large,

$$\begin{aligned} o(1) = \langle J'_{\lambda,\kappa}(u_n), u_n - u\rangle &= a(u_n, u_n - u) + \lambda h_\kappa(u_n)\|u_n\|^2(u_n, u_n - u) \\ &\quad + \frac{\lambda}{2\kappa^2} h'_\kappa(u_n)\|u_n\|^4 (u_n, u_n - u) - \int_{\mathbb{R}^N} f(u_n)(u_n - u)dx \\ &= \left(a + \lambda h_\kappa(u_n)\|u_n\|^2 + \frac{\lambda}{2\kappa^2} h'_\kappa(u_n)\|u_n\|^4 \right)(u_n, u_n - u) + o(1). \end{aligned} \tag{2.9}$$

It forces, as $n \to \infty$,

$$\left(a + \lambda h_\kappa(u)\|u_n\|^2 + \frac{\lambda}{2\kappa^2} h'_\kappa(u_n)\|u_n\|^4 \right)(u_n, u_n - u) = o(1).$$

From the definition of h, we easily obtain $(u_n, u_n - u) \to 0$ and $\|u_n\| \to \|u\|$. Combining this with (2.6), we demonstrate that $u_n \to u$ in H. This finishes the proof. ☐

When $\{u_n\} \subset \mathcal{N}_{\lambda,\kappa}$, using similar procedure of the proof above, we know that result of Lemma 2.7 also holds at level $\widetilde{c}_{\lambda,\kappa}$.

Lemma 2.8. *The $c_{\lambda,\kappa}$ is attained by some $u \in \mathcal{M}_{\lambda,\kappa}$ for λ small, which is a critical point of $J_{\lambda,\kappa}$ in H.*

Proof. Let $\{u_n\} \subset \mathcal{M}_{\lambda,\kappa}$ be such that $J_{\lambda,\kappa}(u_n) \to c_{\lambda,\kappa}$ as $n \to \infty$. By Lemma 2.7, we know that there exists a $u \in H$ such that

$$\begin{aligned} u_n &\to u, \\ u_n^+ &\to v, \\ u_n^- &\to w, \end{aligned} \tag{2.10}$$

in H as $n \to \infty$. Since $u_n \in \mathcal{M}_{\lambda,\kappa}$,

$$a\|u_n^+\|^2 + \lambda h_\kappa(u_n)\|u_n\|^2\|u_n^+\|^2 + \frac{\lambda}{2\kappa^2} h'_\kappa(u_n)\|u_n\|^4\|u_n^2 = \int_{\mathbb{R}^N} f(u_n^+)u_n^+ dx \tag{2.11}$$

Then, by (f_1), (f_2) and Sobolev's inequality, we have

$$\begin{aligned} a\|u_n^+\|^2 - 4\lambda\kappa^4 \le a\|u_n^+\|^2 - \frac{\lambda}{\kappa^2}\|u_n\|^4\|u_n^+\|^2 &\le \varepsilon \int_{\mathbb{R}^N} |u_n^+|^2 dx + C_\varepsilon \int_{\mathbb{R}^N} |u_n^+|^q dx \\ &\le \varepsilon C_1 \|u_n^+\|^2 + C_2\|u_n^+\|^q. \end{aligned}$$

So $\|u_n^+\| \geq C_3 > 0$, similarly, $\|u_n^-\| \geq C_4 > 0$. This implies that $v, w \neq 0$. Since H is a Hilbert space and the project mapping $u \mapsto u^{\pm}$ is continuous in H, we get $u^+ = v$ and $u^- = w$, then $u = u^+ + u^-$ is a sign-changing function. Next we prove $u \in \mathcal{M}_{\lambda,\kappa}$. From $u_n \in \mathcal{M}_{\lambda,\kappa}$, note that

$$\langle J'_{\lambda,\kappa}(u_n), u_n^+\rangle = \langle J'_{\lambda,\kappa}(u_n), u_n^-\rangle = 0,$$

by (2.10) and passing to the limit, we obtain

$$\langle J'_{\lambda,\kappa}(u), u^+\rangle = \langle J'_{\lambda,\kappa}(u), u^-\rangle = 0,$$

which implies $u \in \mathcal{M}_{\lambda,\kappa}$ and $J_{\lambda,\kappa}(u) = c_{\lambda,\kappa}$. Consequently, $J_{\lambda,\kappa}|_{\mathcal{M}_{\lambda,\kappa}}$ attains its minimum at u, then u is a nontrivial critical point of $J_{\lambda,\kappa}$ in $\mathcal{M}_{\lambda,\kappa}$.

It remains to see that u is a critical point of $J_{\lambda,\kappa}$ in H. Because u is a critical point of $J_{\lambda,\kappa}$ in $\mathcal{M}_{\lambda,\kappa}$, we have that $J'_{\lambda,\kappa}(u) = 0$ in $\mathcal{M}_{\lambda,\kappa}$. Moreover, there exists a Lagrange multiplier μ such that

$$J'_{\lambda,\kappa}(u) - \mu\Psi'(u) = 0, \tag{2.12}$$

where $\Psi(u) = \langle J'_{\lambda,\kappa}(u), u\rangle$. It suffices to prove that $\mu = 0$. By (2.12), we have

$$\langle J'_{\lambda,\kappa}(u), v\rangle - \mu\langle\Psi'(u), v\rangle = 0, \quad \text{for any } v \in H. \tag{2.13}$$

Taking $v = u$, we compute that

$$\begin{aligned}
\langle\Psi'(u), u\rangle &= 2a\|u\|^2 + 4\lambda h_\kappa(u)\|u\|^4 + \frac{5\lambda}{\kappa^2}h'_\kappa(u)\|u\|^6 + \frac{\lambda}{\kappa^4}h''_\kappa(u)\|u\|^8 - \int_{\mathbb{R}^N} f'(u)u^2 + f(u)udx \\
&= \lambda\left(2h_\kappa(u)\|u\|^4 + \frac{4}{\kappa^2}h'_\kappa(u)\|u\|^6 + \frac{1}{\kappa^4}h''_\kappa(u)\|u\|^8\right) - \int_{\mathbb{R}^N} f'(u)u^2 - f(u)udx \\
&\leq \lambda\left(8\kappa^4 + 64\kappa^4 + 16\kappa^4 h''_\kappa(u)\right) - \int_{\mathbb{R}^N} f'(u)u^2 - f(u)udx.
\end{aligned}$$

In virtue of (f_4), we know that there exists a positive constant α such that

$$\int_{\mathbb{R}^N} f'(u)u^2 - f(u)udx \geq \alpha > 0.$$

Therefore, $\langle\Psi'(u), u\rangle < 0$ for enough small λ, together with (2.13), it showes that $\mu = 0$. The proof is completed. □

Corollary 2.9. *The $\widetilde{c}_{\lambda,\kappa}$ is attained by some $u \in \mathcal{N}_{\lambda,\kappa}$, which is a critical point of $J_{\lambda,\kappa}$ in H.*

The proof is similar to that of Lemma 2.8, hence it is omitted here.

3 Proof of main results

According to the lemmas and corollaries in Section 2, we easily obtain the following results.

Theorem 3.1. *Assume the conditions (f_1)–(f_4) hold, for λ small, functional $J_{\lambda,\kappa}$ possesses one least energy critical point u_λ which is constant sign and one least energy sign-changing critical point v_λ. Moreover, the energy of the sign-changing critical point is strictly greater than the least energy, that is,*

$$c_{\lambda,\kappa} = J_{\lambda,\kappa}(v_\lambda) > J_{\lambda,\kappa}(u_\lambda) = \widetilde{c}_{\lambda,\kappa} > 0.$$

Proof. By the the lemmas and corollaries in Section 2, we know that $J_{\lambda,\kappa}$ possesses a least energy critical point u_λ and a least energy sign-changing critical point v_λ.

For v_λ^+, in view of the foregoing discussions, there exists a $t = t(v_\lambda^+) > 0$ such that $tv_\lambda^+ \in \mathcal{N}_{\lambda,\kappa}$, then

$$0 < \widetilde{c}_{\lambda,\kappa} = J_{\lambda,\kappa}(u_\lambda) \leq J_{\lambda,\kappa}(tv_\lambda^+) = J_{\lambda,\kappa}(tv_\lambda^+ + 0v_\lambda^-) < J_{\lambda,\kappa}(v_\lambda^+ + v_\lambda^-) = c_{\lambda,\kappa}.$$

Finally, we will prove that u_λ is constant sign. Suppose that u_λ is sign-changing, then $u_\lambda \in \mathcal{M}_{\lambda,\kappa}$ and

$$\widetilde{c}_{\lambda,\kappa} = J_{\lambda,\kappa}(u_\lambda) \geq J_{\lambda,\kappa}(v_\lambda) = c_{\lambda,\kappa} > \widetilde{c}_{\lambda,\kappa},$$

this is absurd. We complete the proof. □

Next we give an important identity to obtain that u_λ and v_λ are bounded uniformly in H. That is a Pohožaev type identity, which was proved in [11, Lemma 2.6], here we omit the details.

Lemma 3.2. *If $u \in H$ is a weak solution of*

$$\left(a + \lambda h_\kappa(u)\|u\|^2 + \frac{\lambda}{2\kappa^2}h'_\kappa(u)\|u\|^4\right)[-\Delta u + bu] = f(u), \quad x \in \mathbb{R}^N, \tag{3.1}$$

then for λ small, the following Pohožaev type identity holds

$$\left(\frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla u|^2dx + \frac{Nb}{2}\int_{\mathbb{R}^N}|u|^2dx\right)\left(a + \lambda h_\kappa(u)\|u\|^2 + \frac{\lambda}{2\kappa^2}h'_\kappa(u)\|u\|^4\right) = N\int_{\mathbb{R}^N}F(u)dx. \tag{3.2}$$

Lemma 3.3. *For u_λ and v_λ obtained in Theorem 3.1, if $\kappa > 0$ is large enough and $\lambda > 0$ is sufficiently small, then u_λ and v_λ are bounded in H, that is, $\|u_\lambda\|, \|v_\lambda\| \leq \kappa$.*

Proof. This result was proved in [11, Lemma 2.7]. However, it plays a key role in proving Theorem 1.1 and for the sake of completeness and convenience to reader, we here give the detail. From $J_{\lambda,\kappa}(v_\lambda) = c_{\lambda,\kappa}$, we also write it as

$$\frac{1}{2}aN\|v_\lambda\|^2 + \frac{1}{4}Nh_\kappa(v_\lambda)\|v_\lambda\|^4 - N\int_{\mathbb{R}^N}F(v_\lambda)dx = c_{\lambda,\kappa}N \tag{3.3}$$

By $J'_{\lambda,\kappa}(v_\lambda) = 0$, we know that (3.2) holds. Combining (3.2) and (3.3), we get that, for λ small,

$$\begin{aligned}\frac{a}{2}\int_{\mathbb{R}^N}|\nabla v_\lambda|^2dx &\leq \left(a + \lambda h_\kappa(v_\lambda)\|v_\lambda\|^2 + \frac{\lambda}{2\kappa^2}h'_\kappa(v_\lambda)\|v_\lambda\|^4\right)\int_{\mathbb{R}^N}|\nabla v_\lambda|^2dx \\ &= c_{\lambda,\kappa}N + \frac{\lambda}{4}Nh_\kappa(v_\lambda)\|v_\lambda\|^4 + \frac{\lambda N}{4\kappa^2}h'_\kappa(v_\lambda)\|v_\lambda\|^6.\end{aligned} \tag{3.4}$$

Now we start to estimate the right hand side of (3.4). As the procedure in the proof of Lemma 2.3, we have, by the definition of h,

$$\begin{aligned}c_{\lambda,\kappa} &\leq J_{\lambda,\kappa}(\varphi + \psi) \\ &= \frac{a}{2}\|\varphi\|^2 + \frac{a}{2}\|\psi\|^2 + \frac{\lambda}{4}h_\kappa(\varphi+\psi)\|\varphi+\psi\|^4 - \int_{\mathbb{R}^N}F(\varphi+\psi)dx \\ &= \frac{a}{2} + \frac{a}{2} - \int_{\mathbb{R}^N}F(\varphi+\psi)dx + \frac{\lambda}{4}h_\kappa(\varphi+\psi)\|\varphi+\psi\|^4 \\ &\leq \frac{a}{2} + \frac{a}{2} - C_1\int_{B_R(0)}\varphi^2dx - C_1\int_{B_R(0)}\psi^2dx + C + \lambda\kappa^4 \\ &\leq C_1 + \lambda\kappa^4.\end{aligned}$$

We also have that

$$\frac{\lambda}{4} N h_\kappa(v_\lambda)\|v_\lambda\|^4 \leq \lambda N \kappa^4,$$

and

$$\frac{\lambda N}{4\kappa^2}\left|h'_\kappa(v_\lambda)\right|\|v_\lambda\|^6 \leq 4\lambda N \kappa^4.$$

Then together with (3.4), we have

$$\frac{a}{2}\int_{\mathbb{R}^N}|\nabla v_\lambda|^2 dx \leq NC_2 + 6\lambda N\kappa^4.$$

Since $J'_{\lambda,\kappa}(v_\lambda) = 0$, we have

$$\begin{aligned} a\|v_\lambda\|^2 + \lambda h_\kappa(v_\lambda)\|v_\lambda\|^4 + \frac{\lambda N}{4\kappa^2} h'_\kappa(v_\lambda)\|v_\lambda\|^6 \\ = \int_{\mathbb{R}^N} f(v_\lambda)v_\lambda dx \leq \varepsilon \int_{\mathbb{R}^N} v_\lambda^2 dx + C_\varepsilon \int_{\mathbb{R}^N} v_\lambda^{2^*} dx. \end{aligned} \tag{3.5}$$

Therefore, by $\mathcal{D}^{1,2}(\mathbb{R}^N) \subset L^{2^*}(\mathbb{R}^N)$ and Sobolev's inequality,

$$\begin{aligned} (a-\varepsilon)\|v_\lambda\|^2 &\leq C_\varepsilon \int_{\mathbb{R}^N} v_\lambda^{2^*} dx - \frac{\lambda N}{4\kappa^2} h'_\kappa(v_\lambda)\|v_\lambda\|^6 \\ &\leq C_3 \int_{\mathbb{R}^N} |\nabla v_\lambda|^2 dx + 8\lambda\kappa^4 \\ &\leq C_4(NC_2 + 6\lambda\kappa^4)^{2^*/2} + 8\lambda\kappa^4. \end{aligned} \tag{3.6}$$

Arguing by contradiction, suppose $\|v_\lambda\| \geq \kappa$. Then, by (3.6), we have

$$\kappa^2 \leq \|v_\lambda\|^2 \leq C_5(NC_2 + 6\lambda\kappa^4)^{2^*/2} + 8C_6\lambda\kappa^4,$$

which is impossible with κ large and λ small. So $\|v_\lambda\| \leq \kappa$, similarly, we get $\|u_\lambda\| \leq \kappa$. The proof is finished. □

In what follows, we start to prove Theorem 1.1.

Proof of Theorem 1.1. Let κ and λ be large and small, respectively. By Theorem 3.1, we know that $J_{\lambda,\kappa}$ possesses a least energy critical point u_λ at level $\widetilde{c}_{\lambda,\kappa}$ and a least energy sigh-changing critical point v_λ at level $c_{\lambda,\kappa}$, and according to Lemma 3.3, we obtain $\|u_\lambda\|, \|v_\lambda\| \leq \kappa$, then $J_{\lambda,\kappa} = J_\lambda$ and u_λ,v_λ are critical point critical of J_λ at level $\widetilde{c}_\lambda$ and c_λ, respectively. Therefore, equation (1.1) has a least energy signed solution u_λ and a least energy sigh-changing solution v_λ.

Finally, we will see the energy of sign-changing solution is strictly more than the least energy. From $J_{\lambda,\kappa} = J_\lambda$ and Theorem 3.1, we have

$$c_\lambda = J_\lambda(v_\lambda) > J_\lambda(u_\lambda) = \widetilde{c}_\lambda > 0.$$

Thus the proof is complete. □

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